**Rossby Waves**

**1. Simple treatment**

We start from the barotropic vorticity equation at the *level of non-divergence*, i.e where ∂ω/∂p=0, which is around 500 mb.



This becomes:



Writing β = ∂f/∂y and assuming constant β (β-plane assumption):



Now we seek solutions to this equation in the form of small perturbations on a background zonal flow, **V** = (U, 0, 0). These perturbations are written (u’, v’, 0), with perturbation vorticity ξ’=(**∇**x**V**)z



This equation contains both ξ’ and v’ and we need a relationship between them. To do this we use the *streamfunction* ψ. Any 2-D velocity field can be expressed as the sum of a non-divergent and an irrotational part:

 **U** = **k**x**∇**ψ + **∇**φ

= (-∂ψ/∂y, ∂ψ/∂x, 0) + (∂ϕ/∂x, ∂ϕ/∂y, 0)

where φ is the potential and ψ the streamfunction; the first term is *nondivergent* because ∇.(**k**x**∇**ψ) ≡ 0, and the second is *irrotational* because **∇**x**∇**ϕ ≡ 0 . In this case we have a level of non-divergence so ∇2ϕ = 0 everywhere (0 = ∇**.U =** ∇**.∇**φ), which means the irrotational wind (**∇**φ) is constant. Thus **U**’ = **k**x**∇**ψ and vorticity



Substituting in the vorticity equation:



This is a homogeneous linear differential equation in ψ, and we look for a wave solution:

ψ = ψ0 exp i(ωt – kx - ℓy)

Since ∇2ψ = -κ2ψ, where κ2 = k2 + ℓ2,



from which we derive the dispersion equation



From this the wave phase speed c < U (since both β and κ2 must be positive) and the difference is greatest for small κ, i.e. long wavelength waves. Indeed, the very longest waves (planetary waves) *retrogress* – they propagate westwards. For synoptic-scale Rossby waves the propagation is generally eastward at a speed slower than the wind – so the air flows though the Rossby wave.

2. **Effect of mean wind varying in y**. This is easily allowed for by writing **U** = $\overline{U}$**(**y) + **U’**(x,y) in the calculation of ξ (z-component of curl **U**).



 So  becomes 

Since the background wind is assumed to be unchanging in time, this means replacing β by β\* where 

The rest of the derivation is the same as in 1, with β\* replacing β.

**3. Group velocity.**

We have seen that the phase velocity c is slower than the mean wind. The group velocity however can be **faster** than the mean wind:



If we set ℓ = 0, so that κ = k, we see that **cg**= (U + β\*/k2 , 0). This would apply for example on a zonal jet stream where β\* > β > 0. In this case new troughs and ridges form downstream faster than the mean wind, even though the phase propagation is slower than the mean wind – i.e. an individual air parcel flows through an individual trough or ridge from west to east, but new troughs and ridges appear ahead of it.

It is possible (e.g. Hoskins and Ambrizzi, JAS 50, 1661-71, 1993) to write:



From the diagram below we see that cos(α) = k/κ, and $\hat{κ}= \left(\frac{k}{κ},\frac{l}{κ}\right)$ is the unit vector normal to the wave crests.

$$\hat{κ}$$

$$\hat{l}$$

α

$$\hat{l}$$

$$\hat{k}$$

Therefore:



In the special case of stationary waves, c=0 and **cg­** is aligned along **κ̂**. Furthermore U=β\*/κ2 in this case so **cg** = 2Ucos(α) **κ̂**. Clearly, for strictly zonal propagation α = 0 and the group propagation velocity is twice the mean wind velocity.

**4. Relaxing the non-divergence requirement.**

The appropriate treatment uses the quasi-geostrophic potential vorticity equation dQ/dt=0, where

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where Φ is the geopotential and σ the static stability parameter in QG theory. QG PV is conserved in isobaric flow, provided the disturbances are small in amplitude (βL/f0 <<1) and the Rossby number is small (U/f0L <<1). L is a length scale for the disturbance.

 If Q=(y)+Q’ then and the derivation proceeds as in 2 with Φ as stream function.